This week

1. Section 9.1: solutions, slope fields, Euler’s method
2. Section 9.2: first-order linear equations
3. Section 9.3: applications
A (first order) differential equation is an equation involving an unknown function and its derivatives
\[ F(x, y, y') = 0 \]

A solution is a function \( y(x) \), that satisfies the differential equation:
\[ F(x, y(x), \frac{dy(x)}{dx}) = 0. \]

A normal (first order) differential equation is an equation of the form
\[ y' = f(x, y) \]

Example:
\[ y' = \cos(x) \]

The function \( y(x) = \sin(x) \) is a solution because
\[ \frac{dy(x)}{dx} = \frac{d}{dx}(\sin(x)) = \cos(x). \]

Every anti-derivative of \( \cos(x) \) is a solution of \( (*) \).

The solutions of \( (*) \) are
\[ y(x) = \sin(x) + C \]
with \( C \) an arbitrary constant.
What is a differential equation?

Example:

\[ y' = 2xy \]

- The function \( y(x) = e^{x^2} \) is a solution because

\[ \frac{d}{dx} y(x) = \]

- For every \( C \) the function \( y(x) = Ce^{x^2} \) is a solution:

\[ \frac{d}{dx} Ce^{x^2} = \]

**Definition**

- An additional condition like \( y(x_0) = y_0 \) where \( x_0 \) and \( y_0 \) are given values is called an initial condition or boundary condition.

- A set of equations of the form

\[
\begin{cases}
  F(x, y, y') = 0, \\
  f(x_0) = y_0,
\end{cases}
\]

is called an initial value problem or boundary value problem.
**Slope fields**

1.5

\[ y(x) \text{ is solution of } y' = f(x, y) \]

passing through \( y_0 = y(x_0) \)

The slope of \( \ell \) is \( y'(x_0) = f(x_0, y_0) \)

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**Slope fields**

1.6

\[ y' = 0 \]

\[ y(x) = \]
Slope fields

1.7

\[ y' = y \]

\[ y(x) = \]

Slopefield of \( y' = y \)

1.8

\[ y' = y - x \]

\[ y(x) = \]

\[ y(0) = 0 : y(x) = \]

\[ y(0) = 1 : y(x) = \]

\[ y(0) = 2 : y(x) = \]

\[ \checkmark \text{Slopefield of } y' = y - x \]
Euler’s method

\[ y' = f(x, y) \]

- Recall that a derivative is the limit of a difference quotient

\[ \frac{dy}{dx} = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h} \]

- For small \( h \) we have

\[ \frac{y(x + h) - y(x)}{h} \approx y'(x) = f(x, y(x)), \]

hence

\[ y(x + h) \approx y(x) + h f(x, y(x)). \]

Euler’s method

- The equation of tangent line \( \ell \) is \( y = y_0 + (x - x_0)f(x_0, y_0) \).

- Approximate \( f(x_0 + h) \) with \( y_0 + hf(x_0, y_0) \).
Euler’s method

\[ \begin{align*} y' &= f(x, y) \\ y(x_0) &= y_0 \end{align*} \]

- Fix the step size \( h \).
- Make a table of points \((x_n, y_n)\), starting with \((x_0, y_0)\), where every point is calculated from the previous one with the equations

\[
\begin{align*}
x_{n+1} &= x_n + h \\
y_{n+1} &= y_n + h f(x_n, y_n)
\end{align*}
\]

\[
\begin{array}{|c|c|c|}
\hline
n & x_n & y_n \\
\hline
0 & 0 & 0 \ \\
1 & 0.5 & 0 \ \\
2 & 1.0 & \ \\
3 & 1.5 & \ \\
4 & 2.0 & \ \\
5 & 2.5 & \ \\
\hline
\end{array}
\]
Euler’s methods

Approximation become better by choosing smaller values for $h$.

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**Definition**

A linear first order differential equation is a differential equation of the form

$$y' + P(x)y = Q(x)$$

where $P$ and $Q$ are functions of $x$.

- Notice that $y' = f(x, y)$ with $f(x, y) = Q(x) - P(x)y$.
- The equation is called first-order because it only contains the first derivative of $y$.
- The equation is called linear because there are no nonlinear terms containing $y$ and $y'$, such as $y^2$ or $\cos(y')$. 
Linear first order differential equations

\[ y' + Py = Q \]

- Assume \( v(x) \) is a function that satisfies the equation
  \[ v' = P v. \]  
  \hspace{1cm} (1)

- Then
  \[ \frac{d}{dx}(vy) = \]

- Integrate left- and right-hand side
  \[ vy = \]

- Divide left- and right-hand side by \( v \):
  \[ y(x) = \]

Linear first order differential equations

\[ y' + Py = Q \]

- Equation (1) is a **separable** differential equation that can be solved by integration (see lectures of week 2):
  \[ v' = Pv \quad \implies \quad v(x) = e^\int P(x) \, dx, \]

  where \( \int P(x) \, dx \) is an anti-derivative of \( P(x) \).

Solving a linear differential equation goes in two steps:

1. Find the **integrating factor** \( v \):
   \[ v(x) = e^\int P(x) \, dx. \]
2. Find the solutions:
   \[ y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx. \]
Always check your answer!

\[ y' + P(x)y = Q(x) \]  

(1)

- For linear first-order differential equations, the solution is always of the form

\[
y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx = g(x) + C h(x).
\]

Check you answer

- The function \( g(x) \) should satisfy equation (1).
- Function \( h(x) \) should satisfy the equation \( y' + P(x)y = 0 \).

The differential equation \( y' + P(x)y = 0 \) is called the complementary equation.

Example 1

\[ y' - 2x y = x \]

- \( P(x) = -2x \) and \( Q(x) = x \).
- \[ \int P(x) \, dx = \]
  hence \( v(x) = \).
- Integrate \( v(x) Q(x) \):
  \[ \int x e^{-x^2} \, dx = \]
- Find \( y \):
  \[ y = \]
Example 2

\[ xy' + 2y = x^3 \quad (x > 0) \]

- Rewrite the equation in the form \( y' + Py = Q \):

\[ y' + \frac{2}{x}y = x^2 \implies P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = x^2 \quad (1) \]

- Calculate the integrating factor:

\[
\int P(x) \, dx =
\]

\[ v(x) = \]

- Find \( y \):

\[
y = \]

RL circuits

\textbf{Inductor} — stores energy in a magnetic field

\textbf{Resistor} — limits the flow of current
4.2 Ohm’s law for RL circuits

\[ L \frac{di}{dt} + Ri(t) = V(t) \]

If we apply a constant voltage \( V(t) = V \) and close the circuit at \( t = 0 \) what will happen with the current \( i(t) \)?

\[ \int P(t) \, dt = \cdot \quad \text{hence } v(t) = \]

4.3

\[ \left\{ \begin{array}{l} \frac{di}{dt} + \frac{R}{L} i(t) = \frac{V}{L}, \\ i(0) = 0. \end{array} \right. \]

Find the general solution:

\[ i(t) = \]

Setting \( i(0) = 0 \) we get \( C = \cdot \quad \text{hence } \]

\[ i(t) = \]
The differential equation

\[
\begin{aligned}
\frac{di}{dt} + \frac{R}{L}i(t) &= \frac{V}{L}, \\
i(0) &= 0.
\end{aligned}
\]

has the solution

\[
i(t) = \frac{V}{R} \left(1 - e^{-\frac{Rt}{L}}\right)
\]

The current will eventually reach a steady state value

\[
i_S = \lim_{t \to \infty} i(t) = \frac{V}{R}.
\]

Step response: it takes time to reach the steady state current \(i_S = V/R\).

- At \(t = L/R\) the current is \((1 - \frac{1}{e})i_S \approx .631i_S\).
- At \(t = 3L/R\) about 95% of the steady state current is reached.
- The steady state is reached faster for smaller values of \(L/R\).
Low-pass filters

Consider a circuit with $R = L = 1$ (to simplify the algebra) and an oscillating voltage source $V(t) = \cos(\omega t)$.

$$\frac{di}{dt} + i(t) = \cos(\omega t)$$

We will show that after a while the solution is

$$i(t) = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)$$

where $\varphi$ is a phase shift that depends on the frequency $\omega$.

With “after a while” we mean that $i(t) \approx \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)$ for large values of $t$. 
Low-pass filters

\[ \frac{di}{dt} + i(t) = \cos(\omega t) \]

- \( \int P(t) \, dt = \int 1 \, dt = t \), hence \( v(t) = e^t \).
- Find the general solution:

\[
\begin{align*}
  i(t) &= \frac{1}{v(t)} \int v(t) \cos(\omega t) \, dt \\
  &= e^{-t} \int e^t \cos(\omega t) \, dt.
\end{align*}
\]

Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \int e^t \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \left( e^t \sin(\omega t) - \int e^t \cdot \omega \cos(\omega t) \, dt \right) \\
= e^t \cos(\omega t) + \omega e^t \sin(\omega t) - \omega^2 \int e^t \cos(\omega t) \, dt.
\]

This gives

\[
\int e^t \cos(\omega t) \, dt = e^t \left[ \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \right] + C.
\]
The general solution is

\[ i(t) = e^{-t} \int e^t \cos(\omega t) \, dt \]

\[ = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) + Ce^{-t}. \]

For large values of \( t \) the term \( Ce^{-t} \) is small, so we may neglect this term:

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t). \]